

Propagation of Forced Sound Waves in Rarefied Gasdynamics

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A kinetic theory description of sound propagation in a simple gas is presented. The results are in very close agreement with experiment through all values of Knudsen number.

LIST OF SYMBOLS

(ρ_0, T_0)	equilibrium density and temperature
$\xi = (\xi_1, \xi_2, \xi_3)$	particle velocity
$\xi = \xi $	particle speed
$f = f(x, t, \xi)$	mass-distribution function of particles
k	Boltzmann's constant
m	particle mass
$R = k/m$	gas constant
$f^0 = \rho_0 / (2\pi RT_0)^{3/2} e^{-\xi^2/2RT_0}$	Maxwellian equilibrium-distribution function
$g = (f - f^0) / f^0$	perturbed distribution function
$U = \xi - \xi_* $	relative speed (asterisk subscript denotes a different value of the velocity; ξ_* is sometimes referred to as the struck particle)
θ, ϵ	angle variables that orient a collision between two particles
$B(\theta, U)$	differential cross section that is also a function of the intermolecular force law

INTRODUCTION

THE description of sound propagation arising from an oscillating piston (forced sound waves) under normal conditions, in a simple gas, is adequately de-

scribed by the classical theory of Stokes and Kirchoff.¹ This theory is, of course, based on the Navier-Stokes equations. However, for high-frequency sound propagation in a low-pressure gas, we can expect the situation to change markedly. The relevant parameter is the Knudsen number (Kn), i.e., the ratio of gas mean free path to sound wavelength (alternately, the ratio of sound frequency to gas collision frequency). To obtain a theory for larger values of Kn one has no recourse but to turn to the Boltzmann equation of kinetic theory. Attempts at the sound problem, based on kinetic theory, have been made by Wang Chang² (Burnett equations) and by Wang Chang and Uhlenbeck³ (super-Burnett equations). However, comparison of these theories with experiment^{4,5} shows that these are at best slight extensions of the Navier-Stokes results. Pekeris and his co-workers^{6,7} using numerical procedures have extended

² C. S. Wang Chang, "On the Dispersion of Sound in Helium," Johns Hopkins Univ. Appl. Phys. Lab. Rept. CM-467, UMH-3-F (1948).

³ C. S. Wang Chang and G. E. Uhlenbeck, "On the Propagation of Sound in Monatomic Gases," Univ. Mich. Eng. Res. Inst. ONR Contr. N60nr-23222 (1952).

⁴ M. Greenspan, *J. Acoust. Soc.* **28**, 644-648 (1956).

⁵ E. Meyer and G. Sessler, *Z. Physik.* **149**, 15-39 (1957).

⁶ C. L. Pekeris, Z. Alterman, and L. Finkelstein, in *Symposium on the Numerical Treatment of Ordinary Differential Equations, Integral, and Intro-Differential Equation of P.I.C.C.* (Birkhauser-Verlag, Basel, 1960), pp. 388-398.

⁷ C. L. Pekeris, Z. Alterman, L. Finkelstein, and K. Frankowski, *Phys. Fluids* **5**, 1608-1616 (1962).

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¹ See, for instance, L. Landau and E. Lifshitz, *Fluid Mechanics* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1959).

this method beyond normal belief. Nevertheless, for Kn of $O(1)$ and larger, the results are quite poor. (For a possible explanation of this, see Sec. IV, comment 4, in the closing section of this paper.)

In this paper, a new kinetic-theory approach, based on kinetic models,⁸⁻¹⁰ is taken. Agreement between the above cited experiments and the present theory is extremely good. An analytical account of this theory, mainly based on the (BGK) (Ref. 8) model, has already been discussed.^{11,12} In the present paper, much more elaborate models are considered and the results of a numerical investigation are also presented. In a companion paper¹³ (henceforth referred to as **I**), the nature of sound waves arising from an initial disturbance (free sound waves) is studied. Since many features and formulas are closely related in the two investigations, frequent reference is made to **I**.

I. GOVERNING EQUATIONS

A general theory of models of the Boltzmann equation was first presented by Gross and Jackson.⁹ Some extensions and modifications were given by Sirovich¹⁰ and a description of this technique suited to the present problem is to be found in **I**. In order to present a coherent account of our work, a brief development of kinetic models is now given.

The one-dimensional linearized Boltzmann equation is¹⁴

$$\left(\frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x}\right)g = \mathcal{L}(g) = \frac{1}{m} \int f^0 [g] B(\theta, U) d\epsilon d\theta d\xi_*, \quad (1)$$

with $[g] = g^* + g' - g_* - g$. This equation is made dimensionless with respect to an as yet unspecified frequency $\nu = 1/\tau$, as follows:

$$t' = \nu t, \quad x' = x\nu/(RT_0)^{1/2}, \quad \xi' = \xi/(RT_0)^{1/2}, \quad B' = \rho_0 \nu B/m. \quad (2)$$

On substituting these into (1) and then removing the repetitious primes, we obtain

$$\left(\frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x}\right)g = L(g) = \int \omega_* [g] B d\theta d\epsilon d\xi_*, \quad (3)$$

⁸ P. F. Bhatnager, E. P. Gross, and M. Krook, *Phys. Rev.* **94**, 511-525 (1954).

⁹ E. P. Gross and E. A. Jackson, *Phys. Fluids* **2**, 432-441 (1959).

¹⁰ L. Sirovich, *Phys. Fluids* **5**, 908-918 (1962).

¹¹ L. Sirovich and J. K. Thurber, "Sound Propagation according to Kinetic Models," New York Univ. Inst. Math. Soc. Rept. AFOSR-1380, MF-17, 1961.

¹² L. Sirovich and J. K. Thurber, in *Rarefied Gas Dynamics*, J. A. Laurmann, Ed. (Academic Press Inc. New York, 1963), pp. 159-180.

¹³ L. Sirovich and J. K. Thurber, in *Rarefied Gas Dynamics* (Academic Press Inc., New York and London to be published). Henceforth referred to as **I**.

¹⁴ See, for instance, S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, London, 1952).

with

$$\omega = e^{-\xi^2/2}/(2\pi)^{1/2}. \quad (4)$$

Employing the following inner product

$$(p, q) = \int \omega p(\xi) q(\xi) d\xi, \quad (5)$$

one easily has that for

$$\psi_{r,l} = \left[S^r L^{l+1} \left(\frac{\xi^2}{2} \right) \xi^l P \left(\frac{\xi_1}{\xi} \right) / \left[\frac{2^{l+1} \Gamma(r+l+(3/2))}{\pi^{1/2} r! (2l+1)} \right] \right]^{1/2}, \quad (6)$$

$$(\psi_{r,l}, \psi_{r',l'}) = \delta_{r,r'} \delta_{l,l'}. \quad (7)$$

S^r and P_l are the Laguerre and Legendre polynomials, respectively.¹⁵ For the case $B = B(\theta)$, satisfied by the Maxwell intermolecular-force law (inverse fifth power), Wang Chang and Uhlenbeck³ showed that the $\psi_{r,l}$ are eigenfunctions of L . It is convenient to reduce the double subscript to a single subscript; i.e., we write

$$r = r(i), \quad l = l(i), \quad \psi_{r,l} = \psi_i, \quad (8)$$

and for the moment leave the precise nature of the transformation, Eq. (8), unspecified.

Expanding g in terms of the ψ_i ,

$$g = \sum_{n=1}^{\infty} a_n \psi_n, \quad (9)$$

and substituting into (3), we have

$$\left(\frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x}\right)g = L(g) = \sum_{n,m=0}^{\infty} a_n \lambda_{nm} \psi_m, \quad (10)$$

where

$$a_n = (\psi_n, g) \quad (11)$$

and

$$\lambda_{nm} = (\psi_n, L(\psi_m)). \quad (12)$$

For Maxwell molecules,

$$\lambda_{nm} = 0, \quad m \neq n. \quad (13)$$

To obtain kinetic models, we write

$$\begin{aligned} L \sim L^N &= \sum_{m,n \leq N} a_n \lambda_{nm} \psi_m + \lambda_{N+1, N+1} \sum_{n=N+1}^{\infty} a_n \psi_n \\ &= \sum_{m,n \leq N} a_n (\lambda_{nm} - \lambda_{NN}) \psi_m + \lambda_{N+1, N+1} g \end{aligned} \quad (14)$$

and substitute into equation (10). The above derivation follows that found in Ref. 10. The method of derivation

¹⁵ See, for instance, W. Magnus and F. Oberheltinger, *Formulas and Theorems for Functions of Mathematical Physics* (Chelsea Publishing Co., New York, 1954).

is essentially due to Gross and Jackson,⁹ who gave it for Maxwell molecules. For $N=3$, the model equation becomes the linearized form of the Krook, or single-relaxation model.⁸

The kinetic-model equation is further simplified by choosing ν of Eq. (2) such that $\lambda_{N+1,N+1}$ (Ref. 16) becomes -1 . The N th-order kinetic model then becomes

$$\left(\frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x} + 1\right)g = \sum_{m, n \leq N} a_n \beta_{nm} \psi_m, \tag{15}$$

where

$$\beta_{nm} = (\lambda_{nm} - \lambda_{N+1,N+1}) / |\lambda_{N+1,N+1}|. \tag{16}$$

II. FORMULATION OF THE PROBLEM

An infinite volume of gas is restricted to the right half-plane. The plane at $x=0$ is oscillating with a frequency ω . In the linearization, suitable boundary conditions are applied at $x=0$. Also, from linearity, the time variation can be taken proportional to

$$e^{i\omega t}. \tag{17}$$

The general form of the model equations in this case is

$$\left(i\omega + \xi_1 \frac{\partial}{\partial x} + 1\right)g = \sum_{n, m \leq N} a_n \beta_{nm} \psi_m. \tag{18}$$

The assumption of (17) avoids the transient due to initial conditions, so that (18), with proper boundary conditions, governs the so-called steady-state problem. The boundary condition at $x=0$ is, in general, of a statistical nature. One writes

$$f(\xi, x=0) = \int_{\xi' \leq 0} K(\xi, \xi') f(\xi', x=0) d\xi', \quad \xi_1 > 0. \tag{19}$$

That is, the outgoing distribution function is given in terms of the incoming distribution function. Only a knowledge of the kernel function $K(\xi, \xi')$ is presumed. In practice, kernel functions giving only diffuse and specular reflections (and their combination) are chosen. For the problem considered in this paper, no particular boundary conditions need be considered. We merely denote formally by $g(x=0) = g_0$ the boundary condition at $x=0$.

Since the gas is in the right half-plane, it is natural to use the Laplace transform. Denoting the Laplace-transform variable by s and using the same symbol for the transformed quantities, (18) becomes

$$(i\omega + s\xi_1 + 1)g = \sum_{m, n \leq N} a_n \beta_{nm} \psi_m + \xi_1 g_0. \tag{20}$$

Under obvious manipulations, this becomes

$$a_k = \sum_{m, n \leq N} a_n \beta_{nm} \left(\psi_k, \frac{\psi_m}{i\omega + s\xi_1 + 1} \right) + \left(\psi_k, \frac{\xi_1 g_0}{i\omega + s\xi_1 + 1} \right). \tag{21}$$

Letting $k=1, \dots, N$, we obtain an $N \times N$ system of linear equations in the a_n . In matrix notation,

$$(1 - c)a = Ma = a^0, \tag{22}$$

where

$$a_k^0 = \left(\psi_k, \frac{\xi_1 g_0}{i\omega + s\xi_1 + 1} \right), \tag{23}$$

$$c_{kn} = \sum_{m \leq N} \beta_{nm} \left(\psi_k, \frac{\psi_m}{i\omega + s\xi_1 + 1} \right).$$

To obtain the solution to the problem, one first inverts the matrix equation (22) and then inverts the Laplace transform. With the exception of an asymptotic analysis, such a plan does not seem feasible at present.

As is well-known, values of s for which M is degenerate are possible plane-wave solutions. In general, a great many of such models of propagation are possible.^{12,17} In this paper, we consider only sound propagation. We therefore wish to compute the zeroes of the dispersion relation,

$$\det(M) = 0, \tag{24}$$

which correspond to sound waves. To do this, it is necessary to know $(\psi_k, [\psi_m / (i\omega + s\xi_1 + 1)])$. A large block of these are given in Ref. 12, and the closed-form representation of these for arbitrary k and m is given in I. These involve the Gaussian and error function of the complex variable $(i\omega + 1)/s$. Aside from an asymptotic analysis, any hand computation involving the dispersion relation soon becomes unfeasible. The dispersion relation was therefore programmed for machine solution. The results of these calculations are described in Sec III.

In order to complete the specification of the problem, it is necessary to give the constants λ_{mn} . These, of course, depend on the gas. More specifically, they depend on the nature of the intermolecular force between gas particles. Two different gas laws are considered: rigid spheres and Maxwell molecules. These represent the limits of hard and soft potentials for a neutral gas. The constants λ_{nl} have been computed, and Tables of these are given in I, for the two gases considered.

Finally, for reasons discussed elsewhere,¹⁷ the order-

¹⁷ An infinite number of modes of propagation are shown to exist for the exact Boltzmann equation in L. Sirovich, Phys. Fluids 6, 10-20 (1963).

¹⁶ All λ_{mm} can be shown to be nonpositive; see Ref. 14.

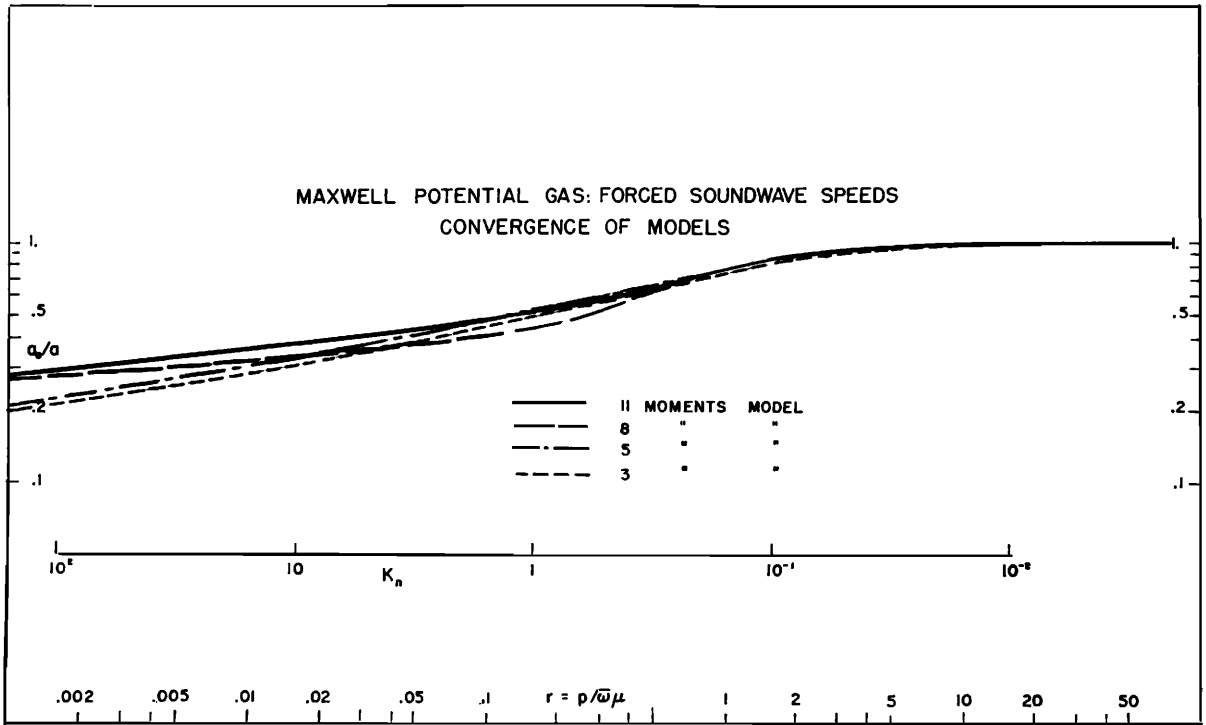


FIG. 1. Maxwell-potential gas: forced sound-wave speeds; convergence of models.

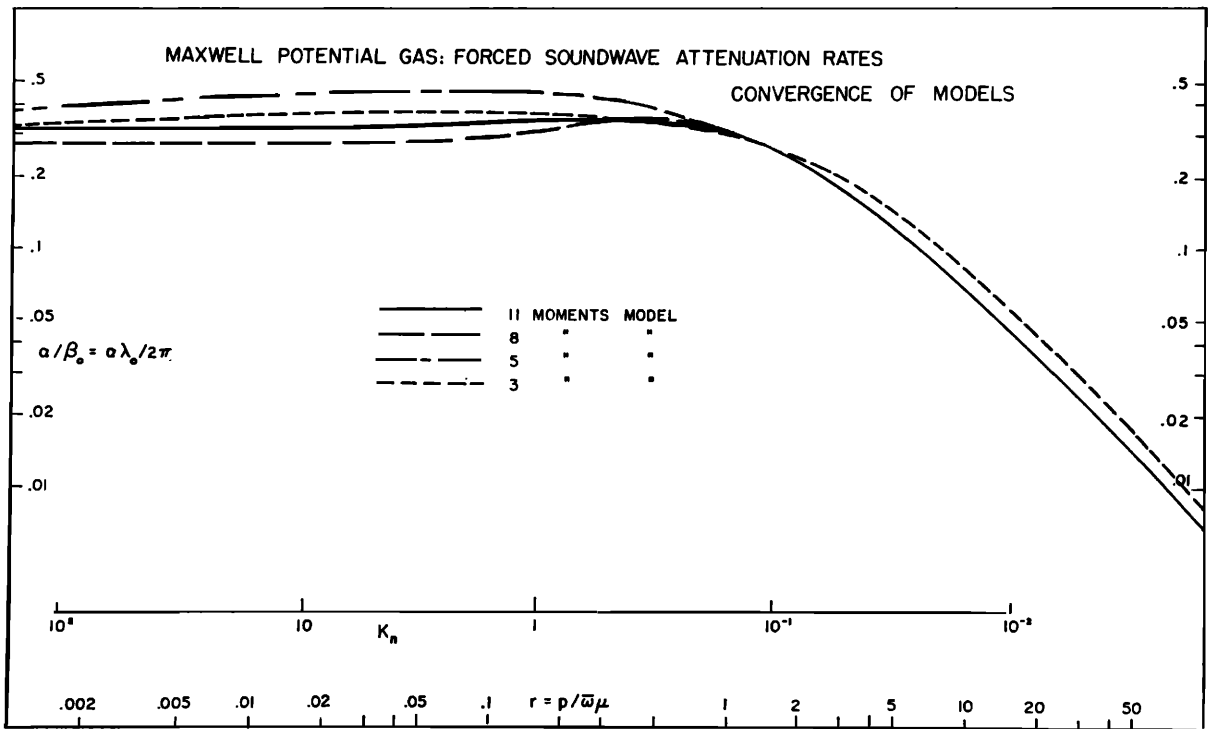


FIG. 2. Maxwell-potential gas: forced sound-wave attenuation rates; convergence of models.

ing, Eq. (8), is taken to correspond to increasing $|\lambda_{mm}|$ (see I for another possibility).

III. DISCUSSION OF THE RESULTS

In presenting the numerical data, we adhere to the description used by Greenspan.⁴ Specifically, we represent the (x,t) dependence of a sound wave by

$$\sim e^{i\bar{\omega}t - \bar{s}x} \tag{25}$$

(corresponding to the Laplace transform of the last section), where $\bar{\omega}$ and \bar{s} are now taken to be the dimensional forms of ω and s . Next writing

$$\bar{s} = \alpha + i\beta, \tag{26}$$

we take for the normalized attenuation rate

$$\alpha/\beta_0 = \alpha a_0 / \bar{\omega} = \alpha \lambda_0 / 2\pi. \tag{27}$$

Based on the adiabatic sound speed a_0 , β_0 is an equivalent wavenumber and λ_0 the corresponding equivalent wavelength. The speed of propagation is

$$a = \bar{\omega} / \beta, \tag{28}$$

and we plot the normalized reciprocal speed

$$a_0/a. \tag{29}$$

Both the speed ratio, Eq. (29), and the attenuation rate, Eq. (27), are functions of a single parameter, which, following Greenspan,⁴ we write as

$$r = p / \bar{\omega}\mu, \tag{30}$$

where p is the pressure and μ the absolute viscosity. To make clear the equivalence of r with the parametrization used earlier, we point out that for Maxwell molecules¹⁴

$$p/\mu = -(\psi_{02}, \mathcal{L}(\psi_{02})) \tag{31}$$

and for rigid spheres¹⁴

$$p/\mu = -(1/1.016)(\psi_{02}, \mathcal{L}(\psi_{02})). \tag{32}$$

\mathcal{L} and ψ_{02} are defined by Eqs. (1) and (6), respectively. The coefficient 1.016 in Eq. (32) arises from the fourth approximation to the viscosity in the sense of Chapman and Cowling.¹⁴

In addition to r , it is convenient for some purposes to also exhibit the associated Knudsen number,

$$\text{Kn} = \frac{\text{MEAN FREE PATH}}{\text{EQUIVALENT WAVELENGTH}}. \tag{33}$$

Basing the mean free path on the rigid-sphere definition,¹⁴ one easily obtains

$$\text{Kn} = 8/r5\pi(2\pi\gamma)^{\frac{1}{2}}, \tag{34}$$

where γ is ratio of specific heats equal to 5/3 in this case.

As already mentioned, the polynomials, ψ_i in (9), are ordered according to increasing values of $|\lambda_{ii}|$. The degeneracy of the eigenvalues of Maxwell molecules¹⁷

then dictates that we take numbers $N=3, 5, 8, 11$. These values of N were taken for both the Maxwell molecule and the rigid spheres. The value $N=11$ allowed the correct determination of μ (Ref. 18) for rigid spheres to within the fourth approximation of viscosity in the sense of Chapman and Cowling.¹⁴

Figure 1 contains the results for the speed of sound of the four kinetic models in the case of Maxwell molecules. In the continuum region ($\text{Kn} < 0.05$), all models except $N=3$ are in excellent agreement. In transition and Knudsen regions¹⁹ [$\text{Kn}=0(1), \text{Kn} > 1$], the curves have spread but there is good agreement between curves. An asymptotic analysis shows that at very high frequency

$$a/a_0 = 0([\ln\omega]^{\frac{1}{2}}). \tag{35}$$

In Fig. 2, the corresponding attenuation rates are plotted. Again, with the exception of the $N=3$ model, all curves fall on each other in the continuum range and show relatively little spread in the transition and Knudsen regions. Each of the attenuation-rate curves has a maximum point. For $N=8$, this occurs relatively sharply at $r \sim 0.5$ while the others exhibit very shallow maxima at $r \sim 0.08$. An asymptotic analysis shows that for all models

$$\alpha/\beta_0 = 0([\ln\omega]^{-\frac{1}{2}}) \tag{36}$$

for very large frequencies. (Therefore, a maximum always exists.)

The failure of the $N=3$ or Krook model to agree with the others has a simple explanation. This model, as Eq. (14) shows,²⁰ has only a single constant. From the manner in which the model equations have been developed, one finds that the Krook model furnishes the heat conduction correctly but the viscosity incorrectly. Specifically, this model leads to a Prandtl number $\text{Pr}=1$. On the other hand, all other models yield the correct value of $\text{Pr}=\frac{2}{3}$ for Maxwell molecules.

The comparable data for the rigid-sphere gas are plotted in Figs. 3 and 4, and the same remarks apply. Figures 5 and 6 compare speed ratios and attenuation rates, respectively, for the Maxwell molecule and rigid-sphere gas when $N=8$ and $N=11$. These two Figures are self-explanatory and, aside from one point, we will not comment on them. On regarding Fig. 6, it is noticed that in the continuum region the rigid sphere and Maxwell-molecule attenuation-rate curves are slightly displaced from one another. Actually, this has been accentuated in the diagram. This divergence arises from the fact that rigid spheres and Maxwell molecules have slightly different Prandtl numbers. As already pointed

¹⁸ For Maxwell molecules, an exact determination of μ is obtained from the eigenvalues.

¹⁹ In using these appellations for the Kn ranges, we are following the customary practice in kinetic theory. But the usual connotations are lost. For instance, $\text{Kn} \gg 1$ does not signify a collisionless gas.

²⁰ One easily can show that $\lambda_{1m}, \lambda_{2m}, \lambda_{3m}$ all vanish; see Ref. 14.

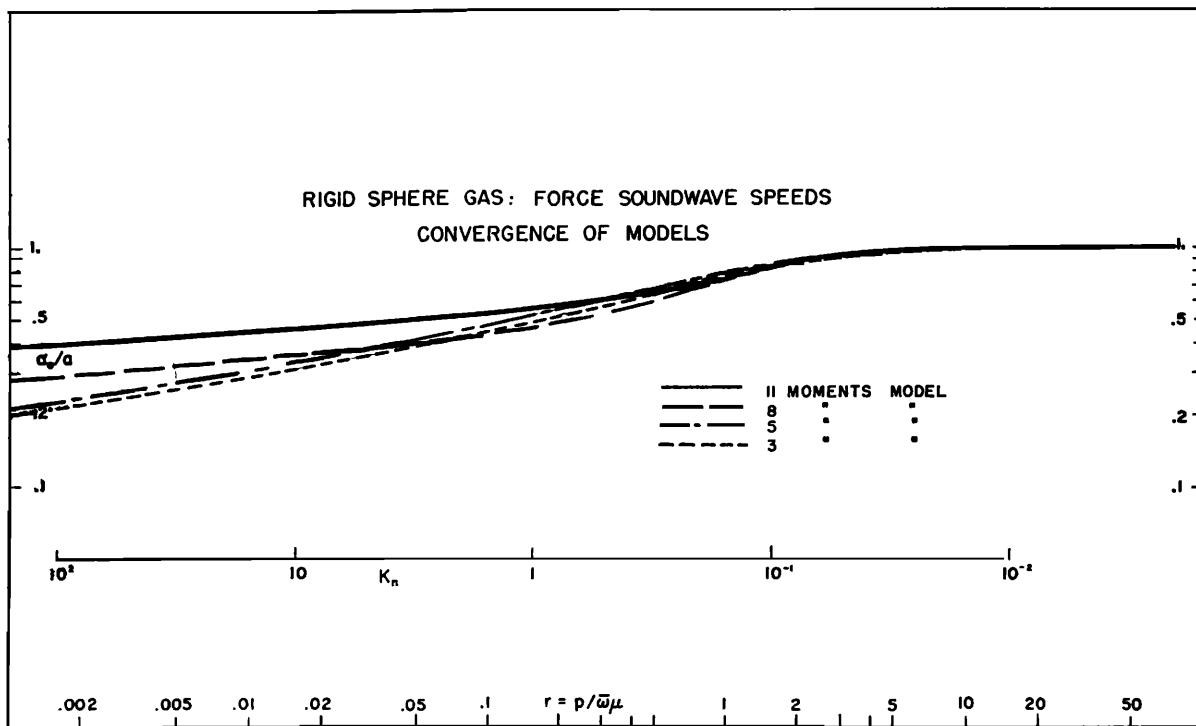


FIG. 3. Rigid-sphere gas: force sound-wave speeds; convergence of models.

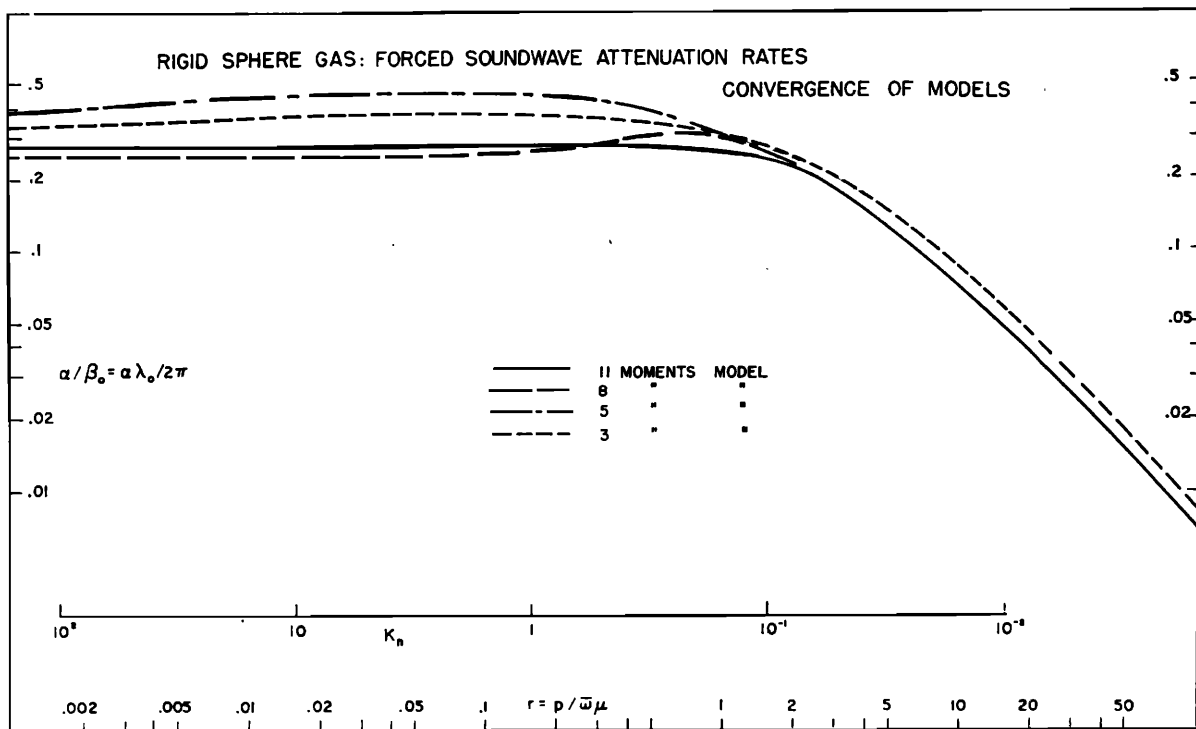


FIG. 4. Rigid-sphere gas: forced sound-wave attenuation rates; convergence of models.

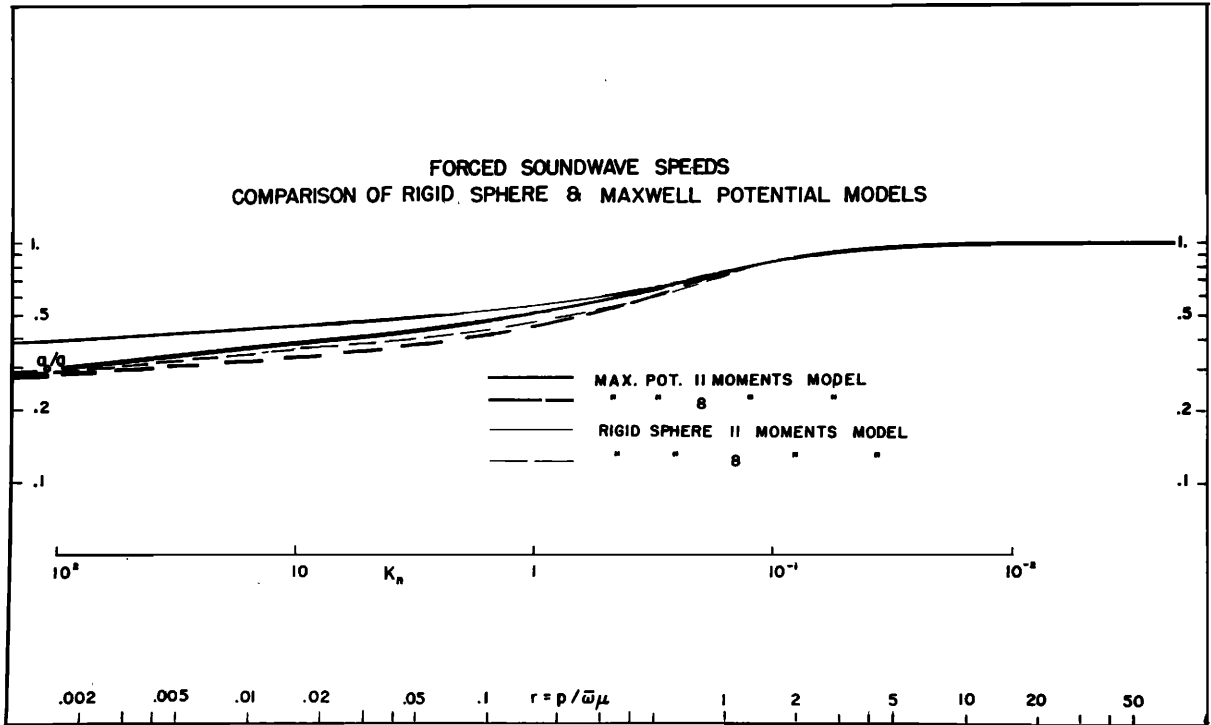


FIG. 5. Forced sound-wave speeds; comparison of rigid sphere and Maxwell-potential models.

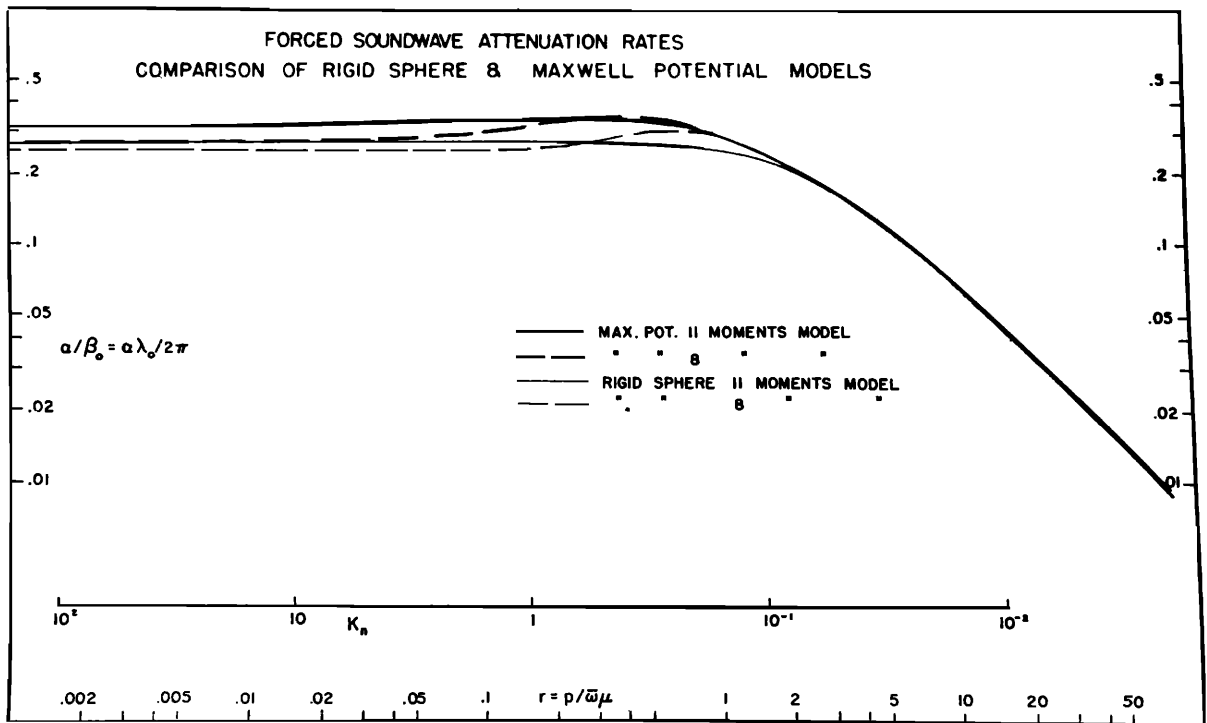


FIG. 6. Forced sound-wave attenuation rates; comparison of rigid sphere and Maxwell-potential models.

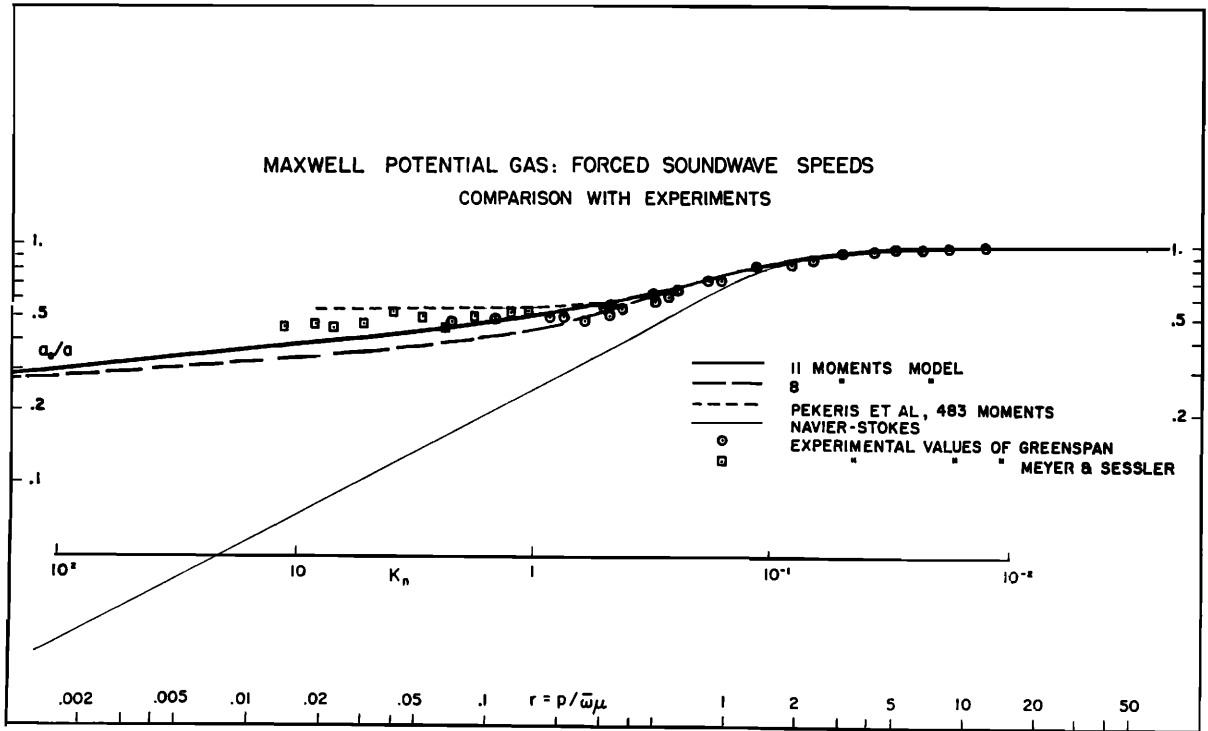


FIG. 7. Maxwell-potential gas: forced sound-wave speeds; comparison with experiments.

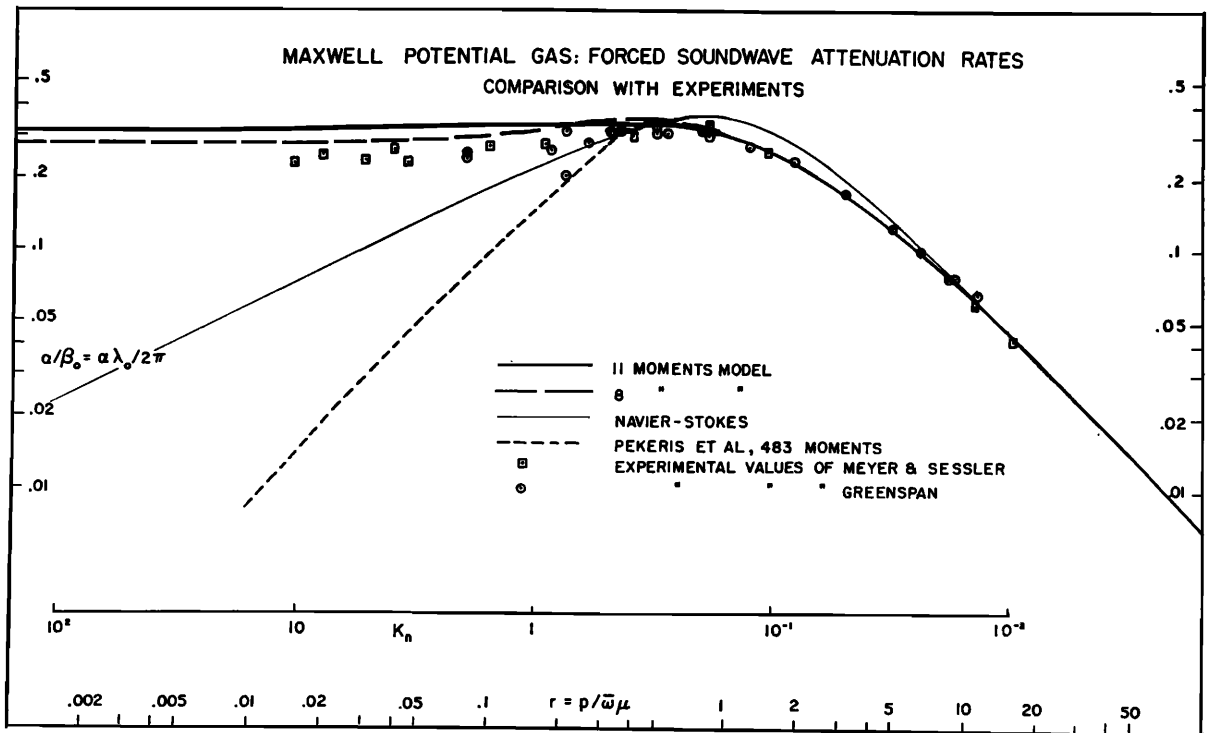


FIG. 8. Maxwell-potential gas: forced sound-wave attenuation rates; comparison with experiments.

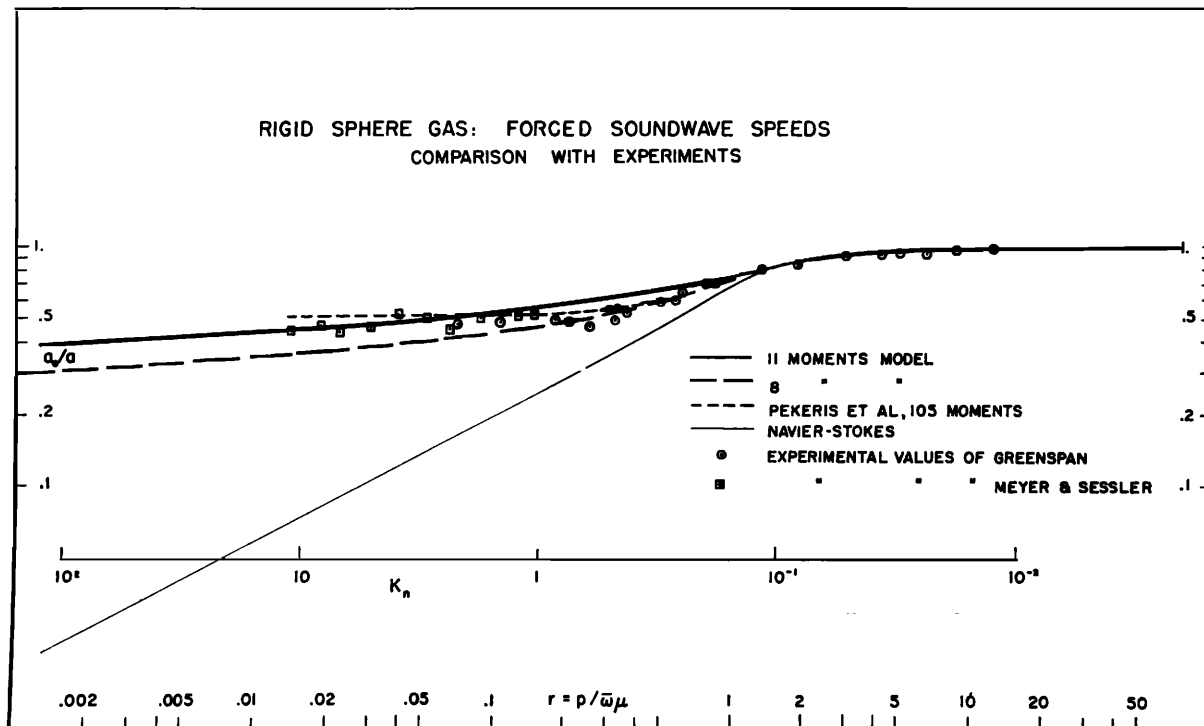


Fig. 9. Rigid-sphere gas: forced sound-wave speeds; comparison with experiments.

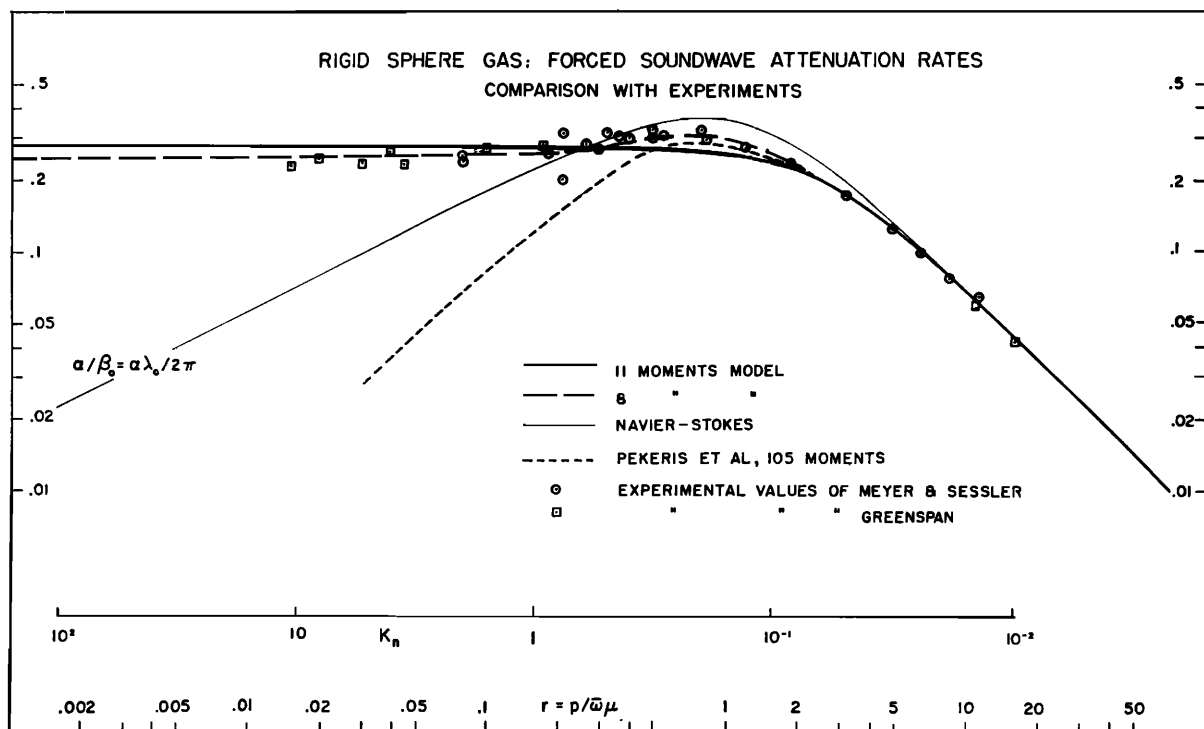


Fig. 10. Rigid-sphere gas: forced sound-wave attenuation rates; comparison with experiments.

out, Maxwell molecules have $Pr = \frac{2}{3}$, whereas for rigid spheres $Pr = 0.661$.

Finally, Figures 7–10 compare the above results with the experiments of Greenspan⁴ and Meyer and Sessler.⁵ Also compared is the Navier–Stokes theory and the work of Pekeris and his coworkers.^{6,7} [The latter is based on the work of Wang Chang and Uhlenbeck³ and is obtained by assuming a finite moment expansion for g in Eq. (9). When substituted into (10), this generates a system of equations in the chosen moments. This can also be regarded as a truncation of the full equation. Pekeris *et al.* numerically solve the derived dispersion relation. By contrast, the method presented here uses the full distribution function g , but with some slight manipulation, Eq. (14), a finite number of moments become distinguished. In referring to either method, it is convenient to refer to the number of moments that each method distinguishes. It is hoped that these remarks will prevent possible confusion.] The experimental values of Greenspan⁴ encompass the five noble gases He, Ne, Ar, Kr, and Xe and the experimental values of Meyer and Sessler⁵ are for Ar. As a rough gauge for comparison between the kinetic theory and the real gases, we note that the Prandtl numbers of the above five gases vary from 0.648 for xenon to 0.674 for neon. This in contrast to abovementioned values of $\frac{2}{3}$ for Maxwell molecules and 0.661 for rigid spheres.

Regarding the speed-ratio plots, Figs. 7 and 9, it can be said that the 8- and 11-moment models are in good agreement with experiment for both Maxwell molecules and rigid spheres. The latter gas model, however, seems decidedly closer to experiment for larger values of Knudsen number. The superiority of rigid spheres in describing the speed ratio is also borne out by the curves of Pekeris *et al.* The latter for rigid spheres falls away from experiment in the Knudsen range. The 11-moment model seems to give the best fit with experiment in this region. As might be expected, the Navier–Stokes speed ratio becomes poor when out of the continuum range.

The attenuation-rate curves, Figs. 8 and 10, offer the widest separation in theoretical prediction. With the exception of the Navier–Stokes curve (which diverges from experiment even in the continuum range), all curves show good agreement with experiment in the continuum as well as part of the transition range. After this, only the kinetic models are in agreement with experiment. Again, the rigid-sphere description is in better agreement and, in fact, the 8-moment rigid sphere is certainly in exceptional agreement with experiment. As in the case with the speed ratio, the 11-moment is at its worst in the transition region, but even here it is less than 10% off the mean experimental values.

IV. REMARKS AND CONCLUSIONS

1. In addition to employing argon, Meyer and Sessler⁵ also used H_2 , a gas with internal degrees of freedom, and air, a complicated mixture of gases. Naturally, the re-

sults found for each of these gases differed from each other. However, in the Knudsen region, strong agreement between all the experimental results, and hence with the present model equations, is found. This indicates that at relatively high frequencies the internal degrees of freedom do not have an opportunity to be excited (the associated time scale is always larger than the time between collisions) and therefore under such conditions these more complicated gases behaved as a simple gas. This suggests that for any gas one should use the appropriate continuum theory such as Navier–Stokes, Burnett, super-Burnett, etc., to give low-frequency data and then match this to the simple gas theory at high frequencies.

2. The results of the kinetic models extend well past the existing experimental values, and indeed sufficient calculations have been obtained to extend the results past those that have been plotted. It seems unlikely, however, that experiments can be performed for such a high Knudsen-number range. The basic difficulty is that a disturbance or oscillation issuing from the wall does not resolve itself into a sound wave until the signal has moved a distance comparable to the mean free path. At distances of less than the mean free path, one has essentially free flow and, hence, in this region one would essentially be measuring the wall distribution function itself. Naturally, since the e -folding distance is small as compared with the mean free path for large Kn, a signal becomes difficult to detect at distances of the order of the mean free path.

3. The fact that rigid spheres gives a finer description than Maxwell molecules probably indicates that a fairly hard intermolecular-force law should be used in the description of the noble gases. This is also borne out by the accepted viscosity–temperature relation for helium²¹ $\mu \sim T^{.647}$. (The Maxwell potential leads to $\mu \sim T$ and rigid spheres to $\mu \sim T^{\frac{1}{2}}$.)

4. The question arises as to why a kinetic model with relatively few distinguished moments is so much better than the results obtained by the 483-moment truncation of Pekeris and his coworkers (see parenthetical remark in Sec. III). First, we remark that clearly such truncations are at most finite Taylor expansions of the exact dispersion relation. Next, Wang Chang and Uhlenbeck³ (see also Ref. 17) showed that, for Maxwell molecules, successive coefficients in the expansion

$$s = \alpha\omega + \beta\omega^2 + i\gamma\omega^3 + \dots \quad (37)$$

for sound waves are given exactly by successive truncations. A study of the consecutively larger truncations used by Pekeris *et al.*^{6,7} shows that for $Kn \ll 1$ closer agreement with experiment is obtained. This implies that the series (37) is not just asymptotic but actually convergent. On the other hand, the wide divergence of the consecutive truncations with experiment for $Kn > 1$

²¹ W. H. Keesom, *Helium* (Elsevier Press, Inc., Amsterdam, 1942).

seems to imply that if the series is convergent it also only has a finite radius of convergence (since a finite Taylor expansion of the dispersion relation does not contain the finite radius of convergence). Simple examples illustrating just this point are easily produced. If the state of affairs is as we concluded, then no truncated system, no matter how large, can hope to give results in the Knudsen (also probably the transition) region. It would therefore seem that this method furnishes extremely precise results in the continuum (and part of the transition) region. Since the intermolecular-force law is at best an approximation, the import of precise results becomes questionable.

5. In Sec. II, we were careful to refer to solutions of the dispersion as giving "possible" plane-wave solutions. Also, we used the Laplace transform rather than the customary normal-mode approach to such problems. This has been done advisedly. One may show that for each kinetic model no plane sound wave exists after some frequency. This critical frequency, however, depends on the particular kinetic models. By increasing N in Eq. (14), the critical frequency is also increased. It therefore depends on the model and, hence, is not physical. As a consequence of using the Laplace transform, the locus of plane-sound-wave modes possesses an analytic continuation. The locus of sound-wave modes for large N is in close agreement with the analytic continuation for lower- N models. This suggests the use of the analytic continuation in predicting sound propagation when past the critical frequency of some particular model. We have done this in presenting the data. The critical values are given in Table I.

A fuller description of this point has been given elsewhere.¹²

6. In a recent report, Kahn and Mintzer²² have also

²² D. Kahn and D. Mintzer (to be published).

TABLE I. Critical frequencies.

N	Maxwell ($p/\bar{\omega}\mu$)	Rigid spheres ($p/\bar{\omega}\mu$)
3	0.701	0.701
5	0.547	0.529
8	0.488	0.415
11	0.413	0.290

considered the sound-propagation problem. In brief, they make a finite polynomial expansion about the free-flow solution. This expansion is substituted into the Boltzmann equation and the corresponding moments are taken. This novel approach has produced some interesting results. As might be expected, their results for $r \ll 1$ are in agreement with the results obtained in the present work. Somewhat surprisingly, however, their results also agree with the continuum results $r \gg 1$. (Some explanation of this is found when one notes that their method is the unsteady analog of the Lee's²³ two-stream theory—where the same circumstances occur.) In the midregion $0.1 < r < 10$, however, the results of Kahn and Mintzer are no longer in close agreement with experiment and the work given here.

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²³ L. Lees "A Kinetic Theory Description of Rarefied Gas Flows," GALCIT Hypersonic Res. Proj. Mem. 51 (1959).